Implying the Risk-Neutral Distribution from the Volatility Skew in Options Pricing

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Introduction

An observation of prices in options markets reveals a departure from the assumptions of the basic Black-Scholes model. Namely, the empirical existence of an implied volatility skew contradicts the Black-Scholes assumption that returns are normally distributed with some constant volatility. Same-dated options on the same underlying exhibit varying implied volatility as a function of strike price. The volatility skew, or the empirical mapping of strike price to implied volatility, is a symptom of a more complicated reality where the risk-neutral distribution of the underlying return is not simply normal.

From the volatility skew, however, we can compute an implied risk-neutral distribution. We use the Black-Scholes formula as a starting point, but we do not accept the Black-Scholes assumptions in the analysis that follows; rather, we simply use the formula as a mapping from volatility space to price space, a common practice in the options trading community, allowing the volatility to vary as function of strike price in order to fully capture the empirical reality of a more sophisticated pricing process.

Solution

Let us call the Black-Scholes price of a European call option $C(\sigma, K)$, where σ is the Black-Scholes volatility and K is the option strike price:

$$C(\sigma(K), K) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{1}$$

where

$$d_2\left(\sigma(K), K\right) = \frac{\ln\left(S_0/K\right) + \left(r - \sigma(K)^2/2\right)T}{\sigma(K)\sqrt{T}} \tag{2}$$

and $d_1 = d_2 + \sigma(K)\sqrt{T}$. We suppress the other inputs, such as the current underlying price (S_0), interest rate (r), and time to expiration (T), in order to avoid cluttering. We then incorporate the volatility skew by expressing σ as a function of the strike price K: $\sigma(K)$. Thus, the price of the call option can be expressed without loss of generality as $C(\sigma(K), K)$. By relaxing the Black-Scholes assumption of fixed volatility, we can fully capture the market prices of same-dated options on the same underlying. Our restrictive assumptions in the analysis that follows are fewer than in Black-Scholes; we require only the existence of reasonably tight bid-ask spreads that define the value of the options and a dense array of strike prices allowing us to reasonably define C as a continuous and differentiable function of K.

First, we observe that the risk-neutral probability distribution can be easily derived from the price of the call options. Consider a call spread on $1/dK$ shares of the underlying consisting of a long position in K-strike calls and a short position in $(K + dK)$-strike calls. This position will have a payoff of zero at expiration if the underlying is below K and a payoff of one at expiration if the underlying is above $K + dK$. As we take dK to zero, this spread becomes tighter and approaches the payoff of a binary option. By a no-arbitrage argument, we determine that the premium for these two portfolios, the call spread and the binary option, should be the same. Also by no-arbitrage, this price is the present-value of the risk-neutral probability that the underlying is above K at expiration. Let us define $F(x)$ as the cumulative density function of the underlying price at expiration in the risk-neutral measure, i.e., the risk-neutral probability that the underlying is less than or equal to x at expiration. Thus, we have

$$\lim_{dK \to 0} \frac{C(\sigma(K), K) - C(\sigma(K+dK), K+dK)}{dK} = -\frac{dC}{dK} = e^{-rT}\left(1 - F(K)\right). \tag{3}$$

Now, using our generalized Black-Scholes formula, we derive another expression for the total derivative of the call option price with respect to the strike price:

$$\frac{dC}{dK} = \frac{\partial C}{\partial \sigma}\frac{d\sigma}{dK} + \frac{\partial C}{\partial K}. \tag{4}$$

The first partial derivative is the Black-Scholes vega that is found directly by differentiating the Black-Scholes formula for the value of a call option:

$$\frac{\partial C}{\partial \sigma} = \nu\left(\sigma(K), K\right) = Ke^{-rT}\phi(d_2)\sqrt{T}, \tag{5}$$

where $\phi(x)$ is the probability density function of the standard normal distribution. The second partial derivative is the negative discounted Black-Scholes risk-neutral probability that the underlying will be in the money at expiration according to the Black-Scholes log-normal distribution:

$$\frac{\partial C}{\partial K} = -e^{-rT}\Phi(d_2), \tag{6}$$

where $\Phi(x) = \int_{-\infty}^{x}\phi(\xi)d\xi$ is the standard normal cumulative density function. This is not to be confused with the true implied underlying distribution $F(x)$. The standard normal functions appear only as an artifact of Black-Scholes and do not represent the implied distribution of the underlying.

Combining equations 3, 4, 5, and 6 above yields the following relation between the implied risk-neutral CDF $F(x)$ and the volatility skew $\sigma(K)$:

$$-e^{-rT}\left(1 - F(K)\right) = Ke^{-rT}\phi(d_2)\sqrt{T}\sigma'(K) - e^{-rT}\Phi(d_2). \tag{7}$$

Explicitly, we have,

$$F(K) = K\phi\left(\frac{\ln\left(S_0/K\right) + \left(r - \sigma(K)^2/2\right)T}{\sigma(K)\sqrt{T}}\right)\sqrt{T}\sigma'(K) - \Phi\left(\frac{\ln\left(S_0/K\right) + \left(r - \sigma(K)^2/2\right)T}{\sigma(K)\sqrt{T}}\right) + 1. \tag{8}$$

Consider the case where the volatility does not vary with K and there is no skew: $\sigma'(K)$ will be equal to zero and the formula reduces to

$$F(K) = 1 - \Phi\left(\frac{\ln\left(S_0/K\right) + \left(r - \sigma(K)^2/2\right)T}{\sigma(K)\sqrt{T}}\right). \tag{9}$$

which is consistent with the Black-Scholes assumption that underlying returns are normally distributed in the risk-neutral measure with expected value r and variance σ^2.

Interpretation

One must be cautious, however, in interpreting this risk-neutral distribution, for the analysis above does not address the question of whether a risk-neutral measure actually exists; rather, it derives a somewhat stylized distribution taking existence for granted. Furthermore, using Black-Scholes formulas while eschewing Black-Scholes assumptions is admittedly unsettling, but once we accept that Black-Scholes is simply a convenient way to convert options prices into comparable terms of annualized volatility, we may again begin to make progress on the problem of options pricing without taking the underpinnings of Black-Scholes too seriously.

Indeed, the relation derived here and the very existence of a volatility skew are testament to a world that is more complicated than Black-Scholes. The insight earned from this calculation is none other than this celebrated empirical fact.

We can observe the implications of this formula when considering the typical volatility smile pattern observed in equity options markets. Let us consider the Black-Scholes CDF as a starting point: $F_{\mathrm{BS}}(x) = 1 - \Phi(d_2(\sigma_{\mathrm{ATM}}, K))$, where $\sigma_{\mathrm{ATM}} = \sigma(S_0 e^{rT})$ is the at-the-money volatility.

The volatility smile characteristic of these markets typically exhibits higher implied volatilities further from the money. The volatility is typically decreasing in strike in the put wing until reaching a minimum around the at-the-money point and typically increasing in strike in the call wing. Thus, we have that $1 - \Phi(d_2(\sigma(K), K)) > F_{\mathrm{BS}}(K)$ for small K and $1 - \Phi(d_2(\sigma(K), K)) < F_{\mathrm{BS}}(K)$ for large K, indicating that the tails of the implied distribution are fatter than suggested by the base log-normal Black-Scholes assumption. Though the term $K\phi(d_2(\sigma(K), K))\sqrt{T}$ has an offsetting effect, it is small compared to the difference between $\Phi(d_2(\sigma(K), K))$ and $\Phi(d_2(\sigma_{\mathrm{ATM}}, K))$.

It is clear, then, that the market consensus of so-called "fat tails," or the relatively higher probability of extreme events as compared to the predictions of the log-normal distribution, is reflected in the volatility smile.

In practice, however, the strike spectrum is not nearly dense enough to easily estimate the derivative of the volatility skew. The availability of this data limits the straight-forward application of equation 8 and effectively defers disputes of the implied risk-neutral probability distribution to the method of volatility skew interpolation.