Probability Density Function in Terms of Moments

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Problem Statement

In this article, we attempt to express the probability density function $f(x)$ of a random variable X in terms of the moments $\mathbb{E}[X^n]$, $n = \{0, 1, 2, \ldots\}$, of the distribution.

Solution

Consider the nth moment of the distribution $f(x)$:

$$\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx.$$

Now consider the Taylor Series expansion of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Thus, the expansion of $e^{-j2\pi xs}$ is

$$e^{-j2\pi xs} = \sum_{n=0}^{\infty} \frac{(-j2\pi xs)^n}{n!}.$$

Now, we consider the following infinite series:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-j2\pi s)^n}{n!} \mathbb{E}[X^n] &= \sum_{n=0}^{\infty} \frac{(-j2\pi s)^n}{n!} \left(\int_{-\infty}^{\infty} x^n f(x) dx \right) \\
&= \int_{-\infty}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(-j2\pi xs)^n}{n!} \right) f(x)\, dx \\
&= \int_{-\infty}^{\infty} e^{-j2\pi xs} f(x)\, dx \\
&= F(s),
\end{aligned}$$

where $F(s)$ is the Fourier transform of $f(x)$. We now have an expression for the Fourier transform of a distribution in terms of its moments:

$$F(s) = \mathcal{F}\{f(x)\}(s) = \sum_{n=0}^{\infty} \frac{(-j2\pi s)^n}{n!} \mathbb{E}[X^n].$$

We recover the probability density function by taking the inverse Fourier transform of this expression:

$$\begin{aligned}
f(x) &= \mathcal{F}^{-1} \left\{ \sum_{n=0}^{\infty} \frac{(-j2\pi s)^n}{n!} \mathbb{E}[X^n] \right\} (x) \\
&= \int_{-\infty}^{\infty} e^{j2\pi xs} \left(\sum_{n=0}^{\infty} \frac{(-j2\pi s)^n}{n!} \mathbb{E}[X^n] \right) ds.
\end{aligned}$$

Example

To verify this result, we apply the formula to the standard normal distribution. The moments of the standard normal distribution are given by

$$\mathbb{E}\left[X^n\right] = \begin{cases} 0 & n \text{ odd} \\ 2^{-n/2}\frac{n!}{(n/2)!} & n \text{ even.} \end{cases}$$

Applying the formula derived above gives

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} e^{j2\pi xs} \left(\sum_{\substack{n \geq 0 \\ n \text{ even}}} \frac{(-j2\pi s)^n}{n!} \left(2^{-n/2} \frac{n!}{(n/2)!} \right) \right) ds \\
&= \int_{-\infty}^{\infty} e^{j2\pi xs} \left(\sum_{m=0}^{\infty} \frac{(-j2\pi s)^{2m} 2^{-m}}{m!} \right) ds \\
&= \int_{-\infty}^{\infty} e^{j2\pi xs} \left(\sum_{m=0}^{\infty} \frac{(-2\pi^2 s^2)^m}{m!} \right) ds \\
&= \int_{-\infty}^{\infty} e^{j2\pi xs} e^{-2\pi^2 s^2} ds \\
&= \mathcal{F}^{-1}\left\{ e^{-2\pi^2 s^2} \right\}(x) \\
&= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\end{aligned}$$

This is in fact the probability density function of the standard normal random variable.