Moments of the Standard Normal Probability Density Function

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We seek a closed-form expression for the $m$th moment of the zero-mean unit-variance normal distribution. That is, given $X \sim \mathcal{N}(0, 1)$, we seek a closed-form expression for $E(X^m)$ in terms of $m$.

First, we note that all odd moments of the standard normal are zero due to the symmetry of the probability density function. Now, we consider the case where $m$ is even. From the definition of expectation, we have

$$E(X^m) = \int_{-\infty}^{\infty} x^m \left( \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{m-1} \left( x e^{-x^2/2} \right) dx$$

We now use integration by parts, taking

$$u = x^{m-1}$$
$$dv = x e^{-x^2/2} dx$$

which gives

$$du = (m-1)x^{m-2}$$
$$v = -e^{-x^2/2}$$

The moment becomes

$$E(X^m) = \frac{1}{\sqrt{2\pi}} \left. \left( -x^{m-1} e^{-x^2/2} \right) \right|_{-\infty}^{\infty} + (m-1) \int_{-\infty}^{\infty} x^{m-2} e^{-x^2/2} dx$$

$$= \frac{m-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{m-2} e^{-x^2/2} dx$$

$$= (m-1)E(X^{m-2})$$

Since $E(X^0) = 1$, the recursive expression can be written as

$$E(X^m) = (m-1)(m-3) \cdots (3)(1)$$

$$= \frac{m!}{\prod_{i=2,4,\ldots,m} i}$$

$$= \frac{m!}{\prod_{i=1}^{m/2} 2i}$$

$$= \frac{m!}{2^{m/2}(m/2)!}$$

In conclusion, for $X \sim \mathcal{N}(0, 1)$, we have that the $m$th moment is

$$E(X^m) = \begin{cases} 0 & m \text{ odd} \\ \frac{2^{-m/2} m!}{(m/2)!} & m \text{ even} \end{cases}$$