Integration by Parts: An Intuitive and Geometric Explanation

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The formula for integration by parts is given below:

$$\int u dv = uv - \int v du \tag{1}$$

While most texts derive this equation from the product rule of differentiation, I propose here a more intuitive derivation for the visually inclined. Let us consider an arbitrary continuous function u of v that exists in the uv-plane:

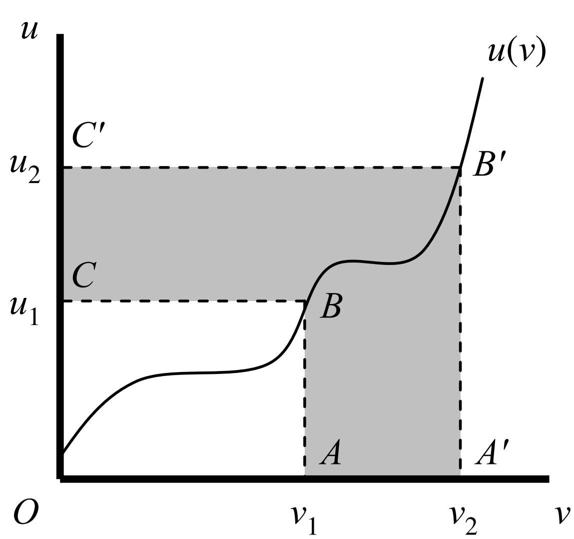

Figure 1: Representation of an arbitrary continuous function in the uv-plane

Considering the geometric interpretation of the integral as the area under a curve, we may express the area of the shape $ABB'A'$ as the integral of the function u with respect to v over the interval (v_1, v_2):

$$\text{Area}\{ABB'A'\} = \int_{v_1}^{v_2} u dv \tag{2}$$

In turn, we can express the area of the shape $CBB'C'$ as the integral of the inverse function $v(u)$ along the u-axis over the interval (u_1, u_2):

$$\text{Area}\{CBB'C'\} = \int_{u_1}^{u_2} v du \tag{3}$$

Now let us express the area of the polygon $CBAA'B'C'$ as the difference in the area of the two rectangles $OA'B'C'$ and $OABC$. The areas of these rectangles are

$$\begin{aligned} \text{Area}\{OABC\} &= u_1 v_1 \\ \text{Area}\{OA'B'C'\} &= u_2 v_2 \end{aligned}$$

Taking the difference of these areas gives the area of the polygon $CBAA'B'C'$:

$$\text{Area}\{CBAA'B'C'\} = \text{Area}\{OA'B'C'\} - \text{Area}\{OABC\} = u_2 v_2 - u_1 v_1 = uv|_{u_1,v_1}^{u_2,v_2} \tag{4}$$

We now consider the expressions from equations (2) and (3) and notice from the geometry of the figure that their sum is the area of $CBAA'B'C'$ in equation (4). Equating (4) to the sum of (2) and (3) gives

$$uv|_{u_1,v_1}^{u_2,v_2} = \int_{v_1}^{v_2} udv + \int_{u_1}^{u_2} vdu \tag{5}$$

This result is merely equation (1) with the terms rearranged. One may find reason for apprehension in the limits of integration in the above. However, we keep in mind that both u and v are ultimately functions of an underlying variable, say, x. The limits in (1) would be expressed in terms of x, and the limits in (5) are in terms of u and v, which vary strictly according to the parameter x along a curve in the uv-plane. Specifically, $u(x_1) = u_1$ as $v(x_1) = v_1$ while $u(x_2) = u_2$ as $v(x_2) = v_2$. With this in mind, we can easily rewrite (5) as

$$uv|_{x=x_1}^{x=x_2} = \int_{x=x_1}^{x=x_2} udv + \int_{x=x_2}^{x=x_2} vdu \tag{6}$$

This is, in fact, the formula for integration by parts. We have sacrificed mathematical rigor for simplicity, but there is at least one concern worth noting. We have drawn $u(v)$ as a one-to-one mapping of the variables u and v. However, we may recall that we have never had to make such assumptions in applying the equation for integration by parts. We argue here without much proof that the analysis above extends to curves that are not necessarily one-to-one, so long as each u and v are strictly functions of the parameter x. The integrals in (6) will still compute, and their sum will reconcile oddities in the shape by allowing for negative areas.