The Musical Chairs Problem

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Problem Statement

There are n passengers who have booked a seat on a full-capacity train ride. They are lined up to board the train, each passenger with a designated seat. Without loss of generality, we say that the nth passenger in line is assigned the nth seat. The first passenger to board, however, is an irrational man with no regard for Kant's categorical imperative. Instead of sitting at his assigned seat, he randomly picks among the n seats. Each subsequent passenger to board, then, first looks to see if his assigned seat is available. If it is, he sits there. Otherwise, he picks randomly from the remaining seats. What is the probability that the last passenger will sit in his assigned seat?

Solution

Let us call the probability that the final passenger sits in his assigned seat p. Also, let us define the useful quantity p_m, which is the probability that the last person will sit in his assigned seat given an m-person subtrain with the initial person picking a random seat. Clearly, $p = p_n$.

Now, consider that when the first person boards, there is a probability $1/n$ that he will pick his own seat, resulting in proper boarding with the final passenger sitting in his assigned seat as well. With probability $1/n$, he will pick the second person's seat, after which the second passenger sees that his seat is taken and must then pick a random seat. In general, with the same probability $1/n$, the first passenger will pick the mth seat, after which boarding will proceed normally until the mth passenger finds that his seat is taken, and he must pick a random seat. Thus, we see that

$$p_n = \frac{1}{n}\left(1 + \sum_{m=2}^{n-1} p_m\right)$$

We establish the relationship

$$np_n = 1 + \sum_{m=2}^{n-1} p_m = p_{n-1} + \left(1 + \sum_{m=2}^{n-2} p_m\right)$$

However,

$$(n-1)p_{n-1} = 1 + \sum_{m=2}^{n-2} p_m$$

allowing us to write

$$np_n = p_{n-1} + (n-1)p_{n-1} = np_{n-1}$$

This implies

$$p_n = p_{n-1} = \cdots = p_2$$

Clearly, $p_2 = \frac{1}{2}$. This is the simple case where there are two seats and the first person randomly picks between his own seat and the seat of the last (second) passenger. Thus, $p = p_n = \frac{1}{2}$. The final passenger will sit in his assigned seat with probability $\frac{1}{2}$.